tions. For optimality, it can be shown that it is necessary that v_h be directed in the direction of the initial primer vector, $\lambda(0)$.

A simple procedure to allow variations in v_h follows:

1) Determine the optimum impulsive trajectory for $v_h = 0$. The initial velocity $\mathbf{v}(0) = \mathbf{v}_d(0)$, the orbital velocity of the departure planet.

2) Solve the impulsive adjoint equations and the ATOM matrix equations. (Analytic solutions for both are known.)

3) Optimize a_0 and v_j.

4) Compute the finite thrust value of $\lambda(0)$.

5) Based on the magnitude of $\lambda(0)$, estimate the optimal value of v_h.

6) Consider the initial velocity to be $v(0) = \mathbf{v}_d(0) + v_h \lambda(0)/p(0)$.

7) Return to step 2 and repeat until convergence is obtained.

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An Approximate Method for Nonlinear Ordinary Differential Equations

TSAI-CHEN SOONG*

The Boeing Company, Seattle, Wash.

A general method is derived for solving problems related to nonlinear, ordinary differential equations with variable coefficients. It consists of subdividing the domain into small regions in which the governing nonlinear differential equation with variable coefficients is reduced to a nonhomogeneous, linear, ordinary differential equation with constant coefficients, one for each region. The solution of the original problem is then taken as the collection of the general solutions of all subdivisions. The arbitrary constants of the complementary solution associated with each region are used to satisfy both the continuity requirements at regional boundaries and the initial or end conditions. The particular solutions compensate for the difference between the desired nonlinear solution and the resulting linear complementary solution. Two versions of the present method are presented. One which is more accurate requires the preparation of a computer program ab initio for each problem treated while the other can be written as a subroutine for general applications. Both versions provide an approximate, continuous solution with accuracy comparable or possibly superior to some commonly used numerical methods. A nonlinear initial-value problem of time-dependent coefficients and a nonlinear deflection of a variable flexible ring were analyzed.

Nomenclature

$A(t)$ = time-dependent cross-sectional area of column
$A_{i,j}$ = deflectional amplitude, Eq. (57)
A_j,B_j = vibratory amplitudes, Eq. (24)
E = Young's modulus
$F(t)$ = deflection function, Eq. (5)
$I(t)$ = time-dependent moment of inertia
I_0 = moment of inertia at $t = 0$, Eq. (9b); also at $s = \pi/2$, Eq. (37)
K_1,K_3 = spring constants of nonlinear foundation, Eq. (53)
L = length of column fixed between ends
P = pulling force on ring, Fig. 2; also axial force, Eq. (53)
$P(t)$ = variable axial compressive load
P_E = Euler buckling load of column at time t
P_0 = initial axial load on column
R = radius of ring
T = period of oscillation, real time, Eq. (36)
a_j,b_j,c_j = constants, Eq. (22)
b = amplitude, Eq. (5); constant related to moment of inertia, Eq. (37)
$c_{i,j}$ = constant coefficients of differential equations, Eq. (55)
$g_{i,j}$ = functions, Eq. (60)
k = constant, Eq. (9a); also Eq. (39)
p = one-half of the pulling force on ring, $P/2$
$p_{i,j}$ = roots, Eq. (59)
$q_{i,j}$ = functions, Eq. (60)
r_0,r_i = outer and inner radii of cylinder, Eq. (34)
s = nondimensional circumferential length of ring, true length/R
$\bar{s}_j$ = midpoint value of s, $(s_j + s_{j-1})/2$
t = time
v = total number of intervals or subdivisions
w = additional deflection of beam-column, Eq. (53)
x = coordinate
y = lateral deflection, Eq. (1); dependent variable, Eq. (62)
β = ratio of inner and outer radii of cylinder, Eq. (13)
δ_h,δ_v = horizontal and vertical displacements of ring, Eqs. (51) and (52)

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* Senior Specialist, Structures Staff, Commercial Airplane Group.

φ = slope of deflection curve, Eq. (38)
ρ = mass density per unit length
ω,ω_0 = frequencies, Eqs. (6) and (10)
τ = nondimensional time, Eq. (11)
τ_0 = nondimensional total burning time (time required to consume whole cylinder)

Superscripts

$(\dot{\ })$ = $d(\ )/dt$
$(\ )'$ = $d(\ )/dx$

Subscripts

i = number of roots of the characteristic equation of a differential equation
j = numbering of intervals or subdivisions
0 = sectional properties at the root of a column or at $t = 0$

I. Introduction

PROBLEMS that can be formulated into nonlinear, ordinary differential equations or partial differential equations reducible to nonlinear ordinary types are encountered very often in theories of elasticity, hydrodynamics, applied mechanics, and other branches of natural sciences. Since we are more familiar with linear algorithms, nonlinear problems are most often studied from the point of view of linear operators. They are usually reduced to linear systems through analytic means or by some asymptotic process that brings them within the realm of functions which have been thoroughly studied by linear methods.

Since only a small number of nonlinear differential equations, mostly in the first- and second-order classes, can be solved exactly, less rigorous methods have to be used in many important practical problems. These methods may be divided into two categories: approximate analytic methods and numerical methods.[1] By approximate analytic methods, we mean analytical procedures for deriving approximate solutions in the form of functions that are close, in some sense, to the unknown exact solutions. In this category, one thinks of the weighted residual methods, iterative methods, and continuous analytical continuation methods.[2] In contrast, the numerical methods, such as the finite-difference methods and Runge-Kutta methods, yield solutions in the form of discrete points along the solution path. The numerical methods, which do not possess continuous solutions, nevertheless have the advantage of a more standardized approach and some, such as the Runge-Kutta methods and several predictor-corrector methods, have been programed into subroutines for computerized solutions of nonlinear, ordinary differential equations. As it stands today, however, no single method is most appropriate to every problem, and the field of research is immense.

The present method makes use of an idea that has been used[3] in the solution of linear, ordinary differential equations with variable coefficients. To describe the method briefly, the domain of the differential equation is divided into small intervals, and it is assumed that when the intervals become infinitesimally small, the variable-coefficient equation may be replaced by an equivalent constant-coefficient equation, a different one for each interval. To extend the method to nonlinear systems, it is further assumed that the errors of replacing the nonlinear part of the nonlinear equation by a finite power series, one in each interval, can be made as small as desired when the intervals are small enough. Consequently, the domain of each interval is governed by a linear, ordinary differential equation with constant coefficient whose general solution consists of a complementary solution of the linear part of the nonlinear differential equation and a particular solution corresponding to the power series representation of the nonlinear part of the nonlinear differential equation. The coefficients in the power series can be determined by the collocation method or the least-mean-square errors for a finite number of terms, and the resulting nonlinear simultaneous equations solved by suitable numerical schemes. Calculations have shown that for all practical purposes only the constant term in the power series is needed which greatly simplifies the analysis. The arbitrary constants of the complementary solution are determined from the results for the previous interval. Thus, calculations are made progressively from one point to the next point along the solution path just as in ordinary numerical methods except that the present solution remains continuous within all the intervals. On the terminal points of the intervals, the solution is continuous for all derivatives up to one order less than the highest degree of the differential equation.

The present solution is exact when the differential equation is a linear, ordinary differential equation with constant coefficients. As the differential equation becomes nonlinear and the coefficients variable, the linear solution is modified in a stepwise manner and converges to the unknown exact solution as the step size decreases indefinitely. Since the solution is expressed in a continuous functional form, it can be operated on directly, such as differentiation and integration, for other purposes. For example, the gross error of not satisfying the differential equation exactly can be estimated by error-squared integration. This cannot be done with ordinary numerical methods.

The intent of the present paper is to introduce the method. Mathematical rigor and exhaustive comparison with other methods are not intended. Applications to an initial-value problem, the nonlinear oscillation of a cylinder with time-dependent cross section, and to a boundary-value problem, the nonlinear deformation of a flexible ring of variable cross section, are given for illustration. Numerical comparisons with the Runge-Kutta method and Hamming's modified predictor-corrector method are given for the first problem. A generalized approach based on the present method that can be programed as a computer subroutine for solving high-order, nonlinear, ordinary differential equations is suggested.

II. Derivations

A. Example of an Initial-Value Problem

The method will first be illustrated by solving an initial-value problem that is reducible to a nonlinear ordinary differential equation with variable coefficients. Assume a column, its cross section is time-dependent. The two ends of the column are kept at a constant distance L. An axial force of constant magnitude P_0 is being applied on the column at all times. In some respects the column resembles a rocket booster made of solid propellant. The lateral vibratory displacement y can be described by the following differential equation with variable coefficients:

$$EI(t)\partial^4 y/\partial x^4 + P(t)\partial^2 y/\partial x^2 + \rho\cdot[A(t)]\partial^2 y/\partial t^2 = 0 \quad (1)$$

where the nonlinearity is introduced by the column compressive load

$$P(t) = P_0 - \frac{EA(t)}{2L}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \quad (2)$$

The end conditions are those assumed for simply supported columns,

$$y(0,t) = \partial^2 y(0,t)/\partial x^2 = 0,\ y(L,t) = \partial^2 y(L,t)/\partial x^2 = 0 \quad (3)$$

and the initial conditions are assumed to be

$$y(x,0) = \sin(\pi x/L),\ \partial y(x,0)/\partial t = 0 \quad (4)$$

Equations (1–4) completely define the given problem.

To solve Eq. (1), let

$$y = y(x,t) = b\sin(\pi x/L)F(t) \quad (5)$$

where b is the amplitude of the midpoint of the column.

Substitution of Eq. (5) into Eqs. (1, 2, and 3), respectively yields

$$\ddot{F} \pm \omega^2 F + 2k^4 F^3 = 0 \quad (6)$$

$$P(t) = P_0 - \rho A(t) L^2 (2k^4/\pi^2) F^2 \quad (7)$$

$$F(0) = 1, \dot{F}(0) = 0 \quad (8)$$

where $(\dot{\ }) = d(\ )/dt$ and

$$2k^4 = Eb^2\pi^4/(4\rho L^4), \quad (9a)$$

$$\omega^2 = \omega(t)^2 = (\pi^2 P_0/\rho A(t) L^2)|P_E I(t)/P_0 I_0 - 1| \quad (9b)$$

The negative sign in Eq. (6) should be used when $P_E I(t)$ is less than $P_0 I_0$ in Eq. (9b). The notation P_E is the Euler column load of the cylinder at time $t = 0$, that is, $P_E = \pi^2 EI_0/L^2$.

When the constant force P_0 is less than the initial Euler load P_E of the cylinder, the column will always vibrate towards it zero-deflection line as long as $P_E I(t)$ is greater than $P_0 I_0$ at time t. The value of ω at $t = 0$ can be obtained from (Eq. 9b) as

$$\omega_0 = \omega(0) = [\pi^2(P_E - P_0)/(\rho A_0 L^2)]^{1/2} \quad (10)$$

The corresponding case of a constant column has been solved exactly by Burgreen[4] and the results will be used to check the accuracy of the present method.

Let τ be the nondimensional time defined as

$$\tau = t\omega_0 \quad (11)$$

For the sake of simplicity, though it is not necessary in the analysis, the rate of decreasing of the volume of the column is assumed to be constant. If it is further assumed that the column is a circular hollow cylinder made entirely of consumable material such as solid propellant, the cross-sectional area of the cylinder which is time-dependent can be expressed simply by

$$A(\tau) = A_0[1 - (\tau/\tau_0)] \quad (12)$$

where τ_0 is the total burning time. If the ratio of the inner radius of the cylinder to its outer radius is denoted by β, the moment of inertia of the cylinder can be written as

$$I(\tau) = I_0\{1 - \beta^4[1 + (\tau/\tau_0)(\beta^{-2} - 1)]^2\}/(1 - \beta^4) \quad (13)$$

From Eqs. (12) and (13), Eq. (9b) can be written as

$$\bar{\omega}^2 = |[f_1(\tau) - (P_0/P_E)f_2(\tau)]/[1 - (P_0/P_E)]| \quad (14)$$

where

$$\bar{\omega} = \omega/\omega_0 \quad (15)$$

$$f_1(\tau) = \frac{1 - \beta^4[1 + (\tau/\tau_0)(\beta^{-2} - 1)]^2}{[1 - (\tau/\tau_0)](1 - \beta^4)} \quad (16a)$$

$$f_2(\tau) = [1 - (\tau/\tau_0)]^{-1} \quad (16b)$$

From Eqs. (10, 11 and 15), the nondimensional form of Eq. (6) becomes

$$d^2F/d\tau^2 \pm \bar{\omega}^2 F + (2k^4/\omega_0^2)F^3 = 0 \quad (17)$$

This is the nonlinear ordinary differential equation with variable coefficients whose solution will be derived by the present method.

First, the variable coefficients in Eq. (17) will be transformed into constants. This is achieved by dividing the independent variable τ into a large number of intervals. It has been shown in Ref. 3 that when the divisions are made sufficiently small, Eq. (17) may be replaced by a set of constant-coefficient differential equations, one for each time interval. In the jth interval, for example, the governing equation is

$$d^2F_j/d\tau^2 \pm \bar{\omega}_j^2 F_j + (2k^4/\omega_0^2)F_j^3 = 0, \quad \tau_{j-1} \leq \tau \leq \tau_j \quad (18)$$

where F_j is valid within the specified time interval and $\bar{\omega}_j$ is a constant associated with the jth interval. According to Ref. 3, the variable coefficients can be reduced to constant values by using the integrated average values, which yields the following value for $\bar{\omega}_j^2$ of Eq. (18)

$$\bar{\omega}_j^2 = \frac{1}{\tau_j - \tau_{j-1}} \int_{\tau_{j-1}}^{\tau_j} \bar{\omega}^2 d\tau = \frac{1}{\tau_j - \tau_{j-1}} \int_{\tau_{j-1}}^{\tau_j} \left| \frac{f_1(\tau) - (P_0/P_E)f_2(\tau)}{1 - (P_0/P_E)} \right| d\tau \quad (19)$$

Or, by simply taking the values at the midpoints which gives

$$\bar{\omega}_j^2 = \bar{\omega}^2|_{\tau = (\tau_j + \tau_{j-1})/2} = \left| \frac{f_1(\bar{\tau}) - (P_0/P_E)f_2(\bar{\tau})}{1 - (P_0/P_E)} \right|_{\bar{\tau} = (\tau_j + \tau_{j-1})/2} \quad (20)$$

where f_1 and f_2 are given in Eqs. (16a) and (16b).

It has been shown[3] that solutions based on using the integrated average values, as given by Eq. (19), are more accurate for the same step size than those based on using the midpoint values, as given by Eq. (20). However, the difference is negligible when the intervals are small enough. If integration is complicated, such as Eq. (19), the simpler approach, as shown in Eq. (20), is recommended.

Equation (18) is now a nonlinear differential equation with constant coefficients whose leading term, the term with the highest order, is linear. We may look upon the nonlinear part of the differential equation as being representable by an infinite power series. However, when the interval between τ_{j-1} and τ_j is small, several terms of the power series might be sufficient.

For the initial state, $f_1(\tau)$ will be larger than $(P_0/P_E)f_2(\tau)$ in Eq. (20) and, consequently, the negative sign in Eq. (18) could be dropped. After transporting the nonlinear term to the right-hand side, one obtains

$$d^2F_j/d\tau^2 + \bar{\omega}j^2 F_j = -(2k^4/\omega_0^2)F_j^3 \quad (21)$$

Equation (21) can be replaced by the following two equations,

$$d^2F_j/d\tau^2 + \bar{\omega}_j^2 F_j = a_j + b_j\tau + c_j\tau^2 + \ldots \quad \tau_{j-1} \leq \tau \leq \tau_j \quad (22)$$

$$-(2k^4/\omega_0^2)F_j^3 = a_j + b_j\tau + c_j\tau^2 + \ldots \quad \tau_{j-1} \leq \tau \leq \tau_j \quad (23)$$

Equation (22) is linear, its general solution, take three terms in the series as example, is

$$F_j = A_j \sin\bar{\omega}_j\tau + B_j \cos\bar{\omega}_j\tau + (1/\bar{\omega}_j^2)(a_j + b_j\tau + c_j\tau^2) - (2c_j/\bar{\omega}_j^4), \quad \tau_{j-1} \leq \tau \leq \tau_j \quad (24)$$

where A_j and B_j are two arbitrary constants of the complementary solution pertaining to the jth interval and a_j, b_j, and c_j belong to the particular solution. Substitution of Eq. (24) into Eq. (23) yields

$$-(2k^4/\omega_0^2)[A_j \sin\bar{\omega}_j\tau + B_j \cos\bar{\omega}_j\tau + (1/\bar{\omega}_j^2)(a_j + b_j\tau + c_j\tau^2) - (2c_j/\bar{\omega}_j^4)]^3 = a_j + b_j\tau + c_j\tau^2, \quad \tau_{j-1} \leq \tau \leq \tau_j \quad (25)$$

Since constants A_j and B_j will be determined by the continuity requirements between the jth interval and the preceding interval, the $(j - 1)$th, they will contain a_j, b_j, and c_j and the corresponding parameters of the $(j - 1)$th interval. Therefore, Eq. (25) contains unknown coefficients a_j, b_j, c_j, and the independent variable τ; the rest of the quantities are known. The values of the three unknowns can be determined by solving three simultaneous, nonlinear algebraic equations obtained by using the Galerkin method over the interval (integration) or by requiring that Eq. (25) be satisfied exactly at three arbitrary points within the interval (point-matching). Newton-Raphson iterative method can be used to solve the simultaneous, nonlinear equation set as follows. Based on the two initial conditions, A_1 and B_1 of the first interval can be expressed as linear functions of a_1,b_1, and c_1. Substitution into Eq. (25) and integration (or by point-matching), one obtains three nonlinear equations which contain a_1,b_1, and c_1 as unknowns. a_1 can be solved easily by temporarily suppressing b_1 and c_1. Then suppressing c_1 only and solve for a_1 and b_1 by using the earlier value of a_1 as a starting value. Finally, a_1,b_1, and c_1 can be solved with the previous a_1 and b_1 as starting values. The first interval is the only interval one may have to solve the a,b, and c values in such stepwise manner. For succeeding intervals after the first, the Newton-Raphson method could rapidly solve the three unknowns after a few iterations because the a,b,c values of the previous interval can be used as starting values.

For problems that require high accuracy with a given step size, one may have to use two or three terms in Eqs. (22) and (23). However, a one-term solution which satisfies Eq. (25) exactly at the midpoint of the interval will be sufficient in practice, with a quality comparable to some existing numerical methods.

In what follows, we shall limit the analysis to a one-term solution and have Eq. (25) satisfied at the midpoint of the interval which yields

$$a_j = (-2k^4/\omega_0^2)F_j^3\big|_{\tau=(\tau_j+\tau_{j-1})/2} = (-2k^4/\omega_0^2)(A_j \sin\bar{\omega}_j\tau + B_j \cos\bar{\omega}_j\tau + a_j/\bar{\omega}_j^2)^3\big|_{\tau=(\tau_j+\tau_{j-1})/2} \quad (26)$$

The corresponding solution for the jth interval is

$$F_j = A_j \sin\bar{\omega}_j\tau + B_j \cos\bar{\omega}_j\tau + (a_j/\bar{\omega}_j^2), \quad \tau_{j-1} \le \tau \le \tau_j \quad (27)$$

Constants A_j and B_j can be expressed in terms of a_j from enforcing the conditions that the function F and its first derivative (since the equation is of second order) should be continuous at $\tau = \tau_{j-1}$, which is the junction between the $(j-1)$th and the jth intervals. Therefore, one obtains

$$F_j = F_{j-1} \text{ and } dF_j/d\tau = dF_{j-1}/d\tau \text{ at } \tau = \tau_{j-1} \quad (28)$$

which, after substitution into Eq. (27), yield

$$A_j = u_{j-1}(1) + u_{j-1}(2)a_j/\bar{\omega}_j^2 \quad (29a)$$

$$B_j = u_{j-1}(3) + u_{j-1}(4)a_j/\bar{\omega}_j^2 \quad (29b)$$

where the known quantities are defined by

$$u_{j-1}(1) = A_{j-1}[\sin\bar{\omega}_{j-1}\tau_{j-1} \sin\bar{\omega}_j\tau_{j-1} + (\bar{\omega}_{j-1}/\bar{\omega}_j) \cos\bar{\omega}_j\tau_{j-1} \cos\bar{\omega}_{j-1}\tau_{j-1}] + B_{j-1}[\sin\bar{\omega}_j\tau_{j-1} \cos\bar{\omega}_{j-1}\tau_{j-1} - (\bar{\omega}_{j-1}/\bar{\omega}_j) \cos\bar{\omega}_j\tau_{j-1} \sin\bar{\omega}_{j-1}\tau_{j-1}] + (a_{j-1}/\bar{\omega}_{j-1}^2) \sin\bar{\omega}_j\tau_{j-1} \quad (30a)$$

$$u_{j-1}(2) = -\sin\bar{\omega}_j\tau_{j-1} \quad (30b)$$

$$u_{j-1}(3) = A_{j-1}[\cos\bar{\omega}_j\tau_{j-1} \sin\bar{\omega}_{j-1}\tau_{j-1} - (\bar{\omega}_{j-1}/\bar{\omega}_j) \sin\bar{\omega}_j\tau_{j-1} \cos\bar{\omega}_{j-1}\tau_{j-1}] + B_{j-1}[\cos\bar{\omega}_j\tau_{j-1} \cos\bar{\omega}_{j-1}\tau_{j-1} + (\bar{\omega}_{j-1}/\bar{\omega}_j) \sin\bar{\omega}_j\tau_{j-1} \sin\bar{\omega}_{j-1}\tau_{j-1}] + (a_{j-1}/\bar{\omega}_{j-1}^2) \cos\bar{\omega}_j\tau_{j-1} \quad (30c)$$

$$u_{j-1}(4) = -\cos\bar{\omega}_j\tau_{j-1} \quad (30d)$$

Substitution of Eqs. (29a) and (29 b) into Eq. (26) yields an algebraic equation to solve for the unknown a_j:

$$a_j = -(2k^4/\omega_0^2)[a_j u_{j-1}(5) + u_{j-1}(6)]^3 \quad (31)$$

where $u_{j-1}(5)$ and $u_{j-1}(6)$ are defined as

$$u_{j-1}(5) = (1/\bar{\omega}_j^2)[1 + u_{j-1}(2) \sin\bar{\omega}_j\tau + u_{j-1}(4) \cos\bar{\omega}_j\tau]_{\tau=(\tau_j+\tau_{j-1})/2} \quad (32)$$

$$u_{j-1}(6) = [u_{j-1}(1) \sin\bar{\omega}_j\tau + u_{j-1}(3) \cos\bar{\omega}_j\tau]_{\tau=(\tau_j+\tau_{j-1})/2}$$

Equation (31), as well as Eq. (26), or Eq. (25) for a three-term solution, might be called, in a novel way, the "characteristic" equation of the solution. It is a departure from the usual methods. One may observe that the right-hand side of Eq. (31) is the nonlinear part of the original nonlinear differential equation and the left-hand side quantity a_j is the designed correction to be added to the linearized solution. We shall see in Sec. III that the idea is extended such that the right-hand side of Eq. (31) includes all the terms, both linear and nonlinear, of the differential equation except the term with the highest order (which should be linear or made linear). Consequently, nonlinear, ordinary differential equations with variable coefficients can be solved in a routine manner after this characteristic equation is established.

Among the multiple roots of Eq. (31), we are interested in one root which is nearest in magnitude to the value of a_{j-1} belonging to the preceding interval. An efficient numerical iterative subroutine should be written to derive this particular a_j closest to the a_{j-1}. Two initial conditions, such as Eqs. (8), and Eq. (26), taken at $\tau = 0$, will be used to solve for the starting values A_0,B_0, and a_0. Afterwards, a_j for all intervals can be derived from Eq. (31). Substitution of a_j into Eqs. (29a) and (29b) yields A_j and B_j for all intervals. An approximate solution of the problem is expressed by Eq. (27) which is continuous and differentiable within the interval and analytically connected to adjacent intervals through Eqs. (28).

In the present formulation of the problem, the negative sign in Eq. (18) is to be used when $f_1(\tau)$ in Eq. (14) is equal to or less than $(P_0/P_E)f_2(\tau)$. Assuming that this first occurs at $\tau = \tau_{j-1}$, then the solution of the second phase of the dynamic response should be written as

$$F_j = A_j \sinh\bar{\omega}_j\tau + B_j \cosh\bar{\omega}_j\tau + (a_j/\bar{\omega}_j^2), \quad \tau_{j-1} \le \tau \le \tau_j \quad (33)$$

Equations (30a–d, 31, and 32) should be changed accordingly from trigonometric functions to hyperbolic functions. Details will not be elaborated here.

A. Numerical Example

Assume that the previous cylinder has the following initial physical properties and loads:

$$\beta = r_i/r_0 = 0.5,\ \omega_0^2/k^4 = 2.0,\ P_0/P_E = 0.5 \quad (34)$$

The initial conditions are given in Eqs. (8). From Eqs. (26) and (27), τ taken at 0, the initial conditions yield the following starting values for the calculations,

$$A_0 = 0,\ B_0 = 1 + (2k^4/\omega_0^2) = 2.0,\ a_0 = -2k^4/\omega_0^2 = -1 \quad (35)$$

Figure 1 shows three deflection-vs-time curves corresponding to columns with total burning time τ_0 equal to $5\omega_0$,$10\omega_0$, and infinity, respectively (curves A, B, and C). From curves A and B, one yields an impression that after the constant axial force P_0 exceeds the column's Euler load which is decreasing monotonously with time, the vibratory motion of the column always terminates, in the practical sense, within one-fourth of its original period. It either diverges rapidly as shown in curve B, if P_0 overtakes the Euler load during the swing-out phase of the motion, or degenerates aperiodically as shown in curve A, if that happens during the swing-in phase of the motion.

It seems that after the axial load P_0 exceeds the Euler load, the ensuing motion of the column is primarily determined by whether, during the period that follows, the momentum of the column is working for the axial compressive force P_0 (curve B) or working against it (curve A). Quantitative physical insights can be obtained by differentiations of F_j to get time histories of velocity, acceleration and shear force of the column. From Burgreen's[4] study of a constant cross sec-

Table 1 Comparison of different numerical methods for calculating the period of a constant cross section column

Size of interval $\Delta\tau/\omega_0$	Errors of calculated period[a]: Runge-Kutta method (based on interpolation)	Hamming's p-c method (based on interpolation)	Method in Sec. II: Interpolation	Method in Sec. II: Eq. (27)	Method in Sec. III: Interpolation	Method in Sec. III: Eq. (27)
0.3	0.0006208	0.0005436	0.0009460	0.0010023	0.002735	0.002804
0.1	−0.0001016	−0.0001122	0.0001307	0.0001307	0.0003376	0.0003376
0.03	−0.0001104	−0.0001106	0.00002956	0.00002956	0.00004768	0.00004768
0.02	−0.0001106	−0.0001106	0.00003242	0.00003242	0.00003719	0.00003815
0.01	−0.0001106				0.00006676	0.00006771

[a] Error = (Calculated value/4.768021) − 1.0.

tion column, the previous results for the variable cross section column seems not unreasonable, even though the constant column does not buckle under all ranges of the axial load.

Burgreen's constant column analysis, which corresponds to the present case in which the burning time is infinity, can be used to check the accuracy of the present method. Burgreen derived the following elliptical integral as the exact period of vibration of a constant column:

$$T = \left[\frac{4}{(2k^4 + \omega_0^2)^{1/2}}\right] \int_0^{\pi/2} \left[1 - \left(\frac{\sin^2\varphi}{[2 + (\omega_0^2/k^4)]}\right)\right]^{-1/2} d\varphi \tag{36}$$

For $\omega_0^2/k^4 = 2$, as given in Eqs. (34), the exact period T, based on Eq. (36), is calculated as $4.768021/\omega_0$.

Two well-known numerical methods were also used to solve the same problem so as to form comparisons with the present method. In Table 1, the errors of the period of vibration of the constant column, compared with the exact value from Eq. (36), are given for the fourth-order integration Runge-Kutta method and Hamming's modified predictor-corrector method. These results were obtained by using the corresponding standard subroutines in the IBM System/360 Scientific Subroutine Package (360A-CM-03X), Version III, in an IBM 360 computer. Since these two numerical methods only yield pointwise solutions, the period is calculated by linear interpolation. The fourth column of Table 1 is the result from the present method by using linear interpolation so as to compare with the other methods on the same basis. However, since the present method provides a continuous solution, the exact period can be found from the intersection of the continuous solution with the zero axis. Such results are shown in the fifth column. The values in these two columns are very close to each other when the size of the interval is small. Calculations were terminated whenever the calculated period converged to the same value as the previous larger step size, or diverged because of computer-orientated errors. The underlined values in the table are recognized as the attainable accuracy based on different methods. As it is, the present method used more residency time in the computer (about 50% more) than the Runge-Kutta method, but the program written for the present method can be further polished if need be.

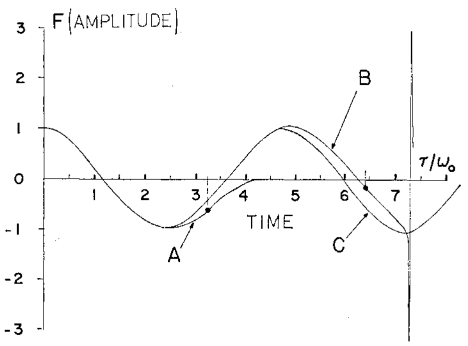


Fig. 1 Amplitude-vs-time curves for the initial-value problem. Curve A: Total burning time $\tau_0 = 5\omega_0$; $P_0I_0 > P_EI(t)$ after $\tau = 3.192\omega_0$; Curve B: Total burning time $\tau_0 = 10\omega_0$; $P_0I_0 > P_EI(t)$ after $\tau = 6.383\omega_0$; Curve C: Total burning time = infinity (constant section column).

The result from Table 1 is too limited to form conclusions. However, one observed that the Runge-Kutta method and Hamming's predictor-corrector method yield better results than the one-term version of the present method for large interval size, while the latter is superior in ultimately achievable accuracy, as interval size decreased. Columns 6 and 7 in Table 1 will be discussed in Sec. III.

B. Example of a Boundary-Value Problem

The second example concerns the large deflection of a flexible ring of variable cross section. Figure 2 shows a free-body diagram of a quarter of the ring that is subjected to a pair of pulling forces of magnitude P acting diametrically on the ring. It is assumed, for simplicity, that the ring section varies according to the following equation:

$$I = I_0(1 + b \cos s), \; 0 \leq s \leq \pi/2 \tag{37}$$

where I_0 is the cross-sectional moment of inertia at $s = \pi/2$. When b is made to zero, the cross section of the ring becomes constant. This case has been solved exactly,[5] and the results will be used to check the accuracy of the present method.

Proceeding as in Ref. 5, the differential equation equating the shear force at a section at coordinate s to the rate of change of the bending moment at that section can be written as

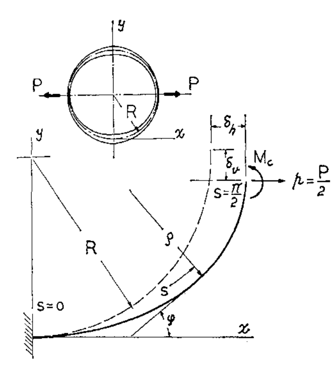
$$(d^2\varphi/ds^2) - b \sin s d\varphi/ds = -(d^2\varphi/ds^2)(b \cos s) - b \sin s + k \sin\varphi \tag{38}$$

where φ is the slope of the deflection curve of the center line of the ring and

$$k = pR^2/EI_0, \; p = P/2 \tag{39}$$

Dividing the length of the quarter ring, which is $\pi/2$, into v equal subdivisions, and representing the right-hand side nonlinear part of the equation as a_j, one obtains, for the jth subdivision, the general soution of the resulting linear differen-

Fig. 2 A free-body diagram of a quarter ring that represents a complete circular ring subjected to diametrical pulling force P.



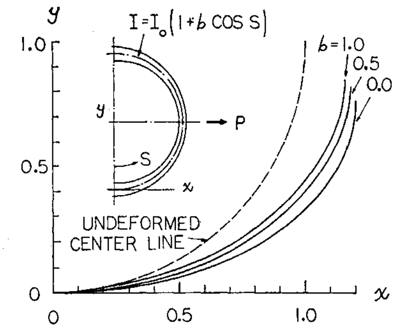


Fig. 3 Deformed shape of the ring ($R = 1$). $k = PR^2/2EI_0 =$ 2.13885.

tial equation and a "characteristic" equation of the present method respectively. They are

$$\varphi_j = A_j + B_j \exp[(b \sin\bar{s}_j)s] - [a_j/(b \sin\bar{s}_j)]s \quad (40)$$

$$s_{j-1} \leq s \leq s_j$$

$$a_j = [-(d^2\varphi_j/ds^2)(b \cos\bar{s}_j) - b \sin\bar{s}_j + k \sin\varphi_j]_{s=\bar{s}_j} \quad (41)$$

$$1 \leq j \leq v$$

where $\bar{s}_j$ can be taken as $(s_j + s_{j-1})/2$ which is the value of s at the midpoint of the jth subdivision.

For a constant ring, $b = 0$, Eqs. (40) and (41) should be replaced by

$$\varphi_j = A_j + B_j s + (a_j/2)s^2, \; s_{j-1} \leq s \leq s_j \quad (42)$$

$$a_j = [k \sin\varphi_j]_{s=\bar{s}_j}, \; 1 \leq j \leq v \quad (43)$$

The continuity at point s_{j-1} requires that φ and $d\varphi/ds$ be equal from both sides of the subdivisions. If we take the ring as of constant cross section for the rest of the analysis, the two equations of continuity are

$$A_j = A_{j-1} - (\tfrac{1}{2})s_{j-1}^2(a_{j-1} - a_j) \quad (44)$$

$$B_j = B_{j-1} + s_{j-1}(a_{j-1} + a_j) \quad (45)$$

Expressed by A_1 and B_1, the constants at the first subdivision, Eqs. (44) and (45) can be written as

$$A_j = A_1 - \frac{1}{2}\sum_{i=1}^{j-1} s_i^2(a_i - a_{i+1}), \; 2 \leq j \leq v \quad (46)$$

$$B_j = B_1 + \sum_{i=1}^{j-1} s_i(a_i - a_{i+1}), \; 2 \leq j \leq v \quad (47)$$

Substitution of Eqs. (46) and (47) into Eqs. (42) and (43) yields

$$\varphi_j = A_1 + sB_1 + \sum_{i=1}^{j-1} s_i(a_i - a_{i+1})(s - \frac{1}{2} s_i) + \frac{1}{2} s^2 a_j, \quad (48)$$

$$j = 1,2, \ldots v$$

$$a_j = k \sin\left[A_1 + \bar{s}_j B_1 + \sum_{i=1}^{j-1} s_i(a_i - a_{i+1})(\bar{s}_j - \frac{1}{2} s_i) + \frac{1}{2} \bar{s}_j^2 a_j\right], \; j = 1,2, \ldots v \quad (49)$$

In the initial-value problem example, calculations started from the values of A_0, B_0, and a_0 which are values corresponding to $t = 0$. In the present example, one may begin from A_1 and B_1, which are the constants corresponding to the first subdivision between $s = 0$ to $s = s_1$. Applying the two boundary conditions $\varphi(s = 0) = 0$ and $\varphi(s = \pi/2) = \pi/2$ to Eq. (48), one obtains

$$A_1 = 0 \quad (50a)$$

$$\frac{\pi}{2} B_1 + \frac{\pi}{4} \sum_{i=1}^{v-1} s_i(a_i - a_{i+1}) + \frac{1}{2} a_v \left(\frac{\pi}{2}\right)^2 = \frac{\pi}{2} \quad (50b)$$

Substituting $A_1 = 0$ into Eq. (49), one can calculate successively the values of a_1, then $a_2, a_3, \ldots$ and finally a_v from Eq. (49), based on an arbitrarily assumed value of B_1. The correct B_1 and its associated set of a_1 to a_v should satisfy the other boundary condition given by Eq. (50b).

The displacements at the loaded end of the quarter ring in Fig. 2 can be calculated either by integration of Eq. (40) which is piecewise integrable, or calculated approximately by using the midpoint values of the slope. For horizontal displacement, the equations are, respectively

$$\delta_h/R = \sum_{j=1}^{v} \int_{s_{j-1}}^{s_j} \cos\varphi_j ds - 1 \quad (51a)$$

$$= \sum_{j=1}^{v} (s_j - s_{j-1}) \cos(\varphi_{j|s=\bar{s}_j}) - 1 \quad (51b)$$

and for vertical displacement, they are

$$\delta_v/R = 1 - \sum_{j=1}^{v} \int_{s_{j-1}}^{s_j} \sin\varphi_j ds \quad (52a)$$

$$= 1 - \sum_{j=1}^{v} (s_j - s_{j-1}) \sin(\varphi_{j|s=\bar{s}_j}) \quad (52b)$$

where φ_j is from Eq. (40).

Figure 3 shows the deflected center lines of a variable cross section, circular ring, calculated from the present method for the cases of $b = 0.5$ and 1.0. The case of a constant ring ($b = 0$) is also shown for reference. The accuracy of the results can be checked for the constant ring since the corresponding exact solution exists.[5] Table 2 shows the correlation with the exact solutions of a constant ring with $k = pR^2/EI_0 = 2.13885$. Its exact horizontal displacement is calculated as $\delta_h = 0.204412R$.

The result as shown in Table 2 seems quite satisfactory. It shows again the excellent degree of accuracy of the one-term solution. Since solution of the boundary-value problem by using numerical methods usually involves guessing correct starting values of parameters at one end to "shoot" for the correct answer at the other end, comparison of merits for different numerical methods by solving the same boundary-value problem can not be entirely objective. Therefore, no comparison will be made for the second example.

It is to be noted that bending moment and shear force at all points of the ring can be obtained by direct differentiations of φ_j of Eq. (40) which is continuous and piecewise differentiable. Also, the right-hand side of Eq. (41), after replacing $\bar{s}_j$ by the variable s, represents the error at point s of the approximate solution. It is an error in the sense that the differential equation, Eq. (38), of which Eq. (41) is its remainder, is satisfied only at the midpoint $\bar{s}_j$ and at no other points within that subdivision. By integrating the square of the difference between the left-hand side and the right-hand side of Eq. (41) with respect to s for all the subdivisions, a kind of "error" can be estimated appropximately for the method. Such features which take advantages of the analytic properties of the present method are not available in some current numerical methods, such as the Runge-Kutta method.

Table 2 Correlation with constant cross section circular ring

Number of subdivisions (Total length $= \pi/2$)	Error of calculated horizontal displacement[a]
10	+0.01068
31	+0.001104
98	+0.0001118
157	+0.00004445
314	+0.00001197
392	+0.000008029

[a] Error = [Value calculated from Eq. (51b)/0.204412] − 1.

C. Higher Order Boundary-Value Problems

For systems with higher order than the second, the initial-value problems should present no difficulty and consequently only boundary-value problems will be discussed briefly here. Take for example, the differential equation for the deflection of a finite column of variable cross section resting on a nonlinear softening foundation,

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w}{dx^2}\right) + P\frac{d^2w}{dx^2} + K_1w - K_3w^3 = -P\frac{d^2\bar{w}(x)}{dx^2} \quad (53)$$

where $\bar{w}(x)$ is the given initial deflection, w is the additional deflection due to the axial load P on the column, and K_1 and K_3 are, respectively, the linear and nonlinear spring constants of the foundation. The buckling case for an infinite column of constant cross section with random initial deflection has been solved by Fraser and Budiansky.[6] Equation (53) can be expanded to

$$f_1(x)w'''' + f_2(x)w''' + [f_3(x) + P]w'' + K_1w = K_3w^3 + f_4(x)P \quad (54)$$

where $f_1(x)$ to $f_4(x)$ are functions of x and can be calculated when $I(x)$ and $\bar{w}(x)$ are given. The notation $(\ )'$ denotes $d(\ )/dx$.

As before, the reduced differential equation for the jth subdivision with $j = v$ as the maximum can be written as

$$c_{1,j}w_j'''' + c_{2,j}w''' + (c_{3,j} + P)w_j'' + K_1w_j = K_3w_j^3 + c_{4,j}P \quad x_{j-1} \le x \le x_j \quad (55)$$

where the constant coefficients are midpoint values defined by

$$c_{i,j} = f_i(\bar{x}_j)|_{\bar{x}_j = \frac{1}{2}(x_j + x_{j-1})}, \ i = 1,2,3,4 \quad (56)$$

As before, when the right-hand side of Eq. (55) is replaced by a constant a_j, the general solution of the resulting linear differential equation and the associated characteristic equation of the present method are obtained as

$$w_j = \text{Real part of}\left(\sum_{i=1}^{4} A_{i,j}\exp(p_{i,j}x) + (a_j/K_1)\right) \quad (57)$$
$$x_{j-1} \le x \le x_j$$

$$a_j = (K_3w_j^3 + c_{4,j}P)|_{x=\bar{x}_j=\frac{1}{2}(x_j+x_{j-1})} \quad (58)$$

where $A_{i,j}$ and $p_{i,j}$ are complex quantities. The four $p_{i,j}$ are the four roots of the following polynomial, which is obtained by substituting a typical term of w_j into the linear homogeneous part of Eq. (55),

$$c_{1,j}p^4 + c_{2,j}p^3 + (c_{3,j} + P)p^2 + K_1 = 0 \quad (59)$$

By matching w and its three consecutive derivatives at x_{j-1}, which is the terminal between the $(j-1)$th and the jth subdivisions, one obtains four simultaneous equations the solutions of which can be written as

$$A_{i,j} = g_{i,j} + q_{i,j}(a_j/K_1), \ i = 1,2,3,4 \quad (60)$$

where $g_{i,j}$ are functions of $A_{1,j-1}$ to $A_{4,j-1}$ and a_{j-1}, which have been calculated from the $(j-1)$th subdivision. $q_{i,j}$ is a function of $p_{i,j}$. Consequently, $A_{i,j}$ is linearly proportional to a_j.

Substitution of Eqs. (57) and (60) into Eq. (58) yields

$$a_j = K_3\left[\frac{a_j}{K_1}\left(1 + \sum_{i=1}^{4} q_{i,j}e^{p_{i,j}\bar{x}_j}\right) + \sum_{i=1}^{4} g_{i,j}e^{p_{i,j}\bar{x}_j}\right]^3 + c_{4,j}P \quad (61)$$

where $\bar{x}_j = \frac{1}{2}(x_j + x_{j-1})$.

For a fourth-order boundary-value problem such as this one, there are only two conditions at each end. Consequently, two A's among the four A's have to be assumed arbitrarily to start the calculation, then a_j for each subdivision is solved from Eq. (61) numerically. The correct starting values of A's should finally satisfy the two end conditions at the other end. The numerical manipulations are not difficult. An example of the iteration technique to solve a set of five nonlinear simultaneous equations has been given by Kempner in a postbuckling analysis of shells.[7]

III. A Generalized Approach

The method illustrated in Sec. II may be extended to entirely computerized applications. Assuming that a nonlinear, ordinary differential equation with variable coefficients has been reduced to a nonlinear, ordinary differential equation with constant coefficients by substituting the independent variable x, which appears in the coefficients, for $\bar{x}_j$, where $\bar{x}_j = \frac{1}{2}(x_j + x_{j-1})$. Then the differential equation, say a fourth-order equation, corresponding to the jth subdivision, can be written as

$$y_j'''' = f(y_j, y_j', y_j'', y_j'''), \ x_{j-1} \le x \le x_j \quad (62)$$

where $(\ )' = d(\ )/dx$.

When the complete right-hand side of Eq. (62) is replaced by a constant a_j, the general solution of the resulting linear equation, $y_j'''' = a_j$, and the characteristic equation can be written respectively as

$$y_j = A_j + B_jx + C_jx^2 + D_jx^3 + (a_j/24)x^4 \quad (63)$$
$$x_{j-1} \le x \le x_j$$

$$a_j = f(y_j, y_j', y_j'', y_j''')|_{x=\bar{x}_j=\frac{1}{2}(x_j+x_{j-1})} \quad (64)$$

The requirements that y_j and its first three consecutive derivatives should be continuous at x_{j-1}, which is the terminal between the jth and its preceding subdivision, yield four simultaneous equations. The solutions of which are

$$\begin{bmatrix} A_j \\ B_j \\ C_j \\ D_j \end{bmatrix} = \begin{bmatrix} A_{j-1} \\ B_{j-1} \\ C_{j-1} \\ D_{j-1} \end{bmatrix} + \begin{bmatrix} -x_{j-1}^4/24 \\ x_{j-1}^3/6 \\ -x_{j-1}^2/4 \\ x_{j-1}/6 \end{bmatrix}(a_{j-1} - a_j) \quad (65)$$

Substitution of Eqs. (65) into Eq. (63) and its derivatives yields

$$\begin{aligned} x_j|_{x=\bar{x}_j} = {} & A_{j-1} - (x_{j-1}^4/24)(a_{j-1} - a_j) + \\ & [B_{j-1} + (x_{j-1}^3/6)(a_{j-1} - a_j)]\bar{x}_j + \\ & [C_{j-1} - (x_{j-1}^2/4)(a_{j-1} - a_j)]\bar{x}_j^2 + \\ & [D_{j-1} + (x_{j-1}/6)(a_{j-1} - a_j)]\bar{x}_j^3 + (a_j/24)\bar{x}_j^4 \end{aligned} \quad (66a)$$

$$\begin{aligned} y_j'|_{x=\bar{x}_j} = {} & B_{j-1} + (x_{j-1}^3/6)(a_{j-1} - a_j) + \\ & 2[C_{j-1} - (x_{j-1}^2/4)(a_{j-1} - a_j)]\bar{x}_j + \\ & 3[D_{j-1} + (x_{j-1}/6)(a_{j-1} - a_j)]\bar{x}_j^2 + (a_j/6)\bar{x}_j^3 \end{aligned} \quad (66b)$$

$$\begin{aligned} y_j''|_{x=\bar{x}_j} = {} & 2[C_{j-1} - (x_{j-1}^2/4)(a_{j-1} - a_j)] + \\ & 6[D_{j-1} + (x_{j-1}/6)(a_{j-1} - a_j)]\bar{x}_j + (a_j/2)\bar{x}_j^2 \end{aligned} \quad (66c)$$

$$y_j'''|_{x=\bar{x}_j} = 6[D_{j-1} + (x_{j-1}/6)(a_{j-1} - a_j)] + a_j\bar{x}_j \quad (66d)$$

Let $y(0)$, $y'(0)$, $y''(0)$ and $y'''(0)$ be the initial conditions given, the starting values of A to D and a_0 can be found from Eqs. (66a–d) and (64) as

$$A_0 = y(0), \ B_0 = y'(0), \ C_0 = y''(0)/2, \ D_0 = y'''(0)/6 \quad (67)$$

$$a_0 = f(y = y(0), y' = y'(0), y'' = y''(0), y''' = y'''(0)) \quad (68)$$

Based on Eqs. (67) and (68) to start calculation, and Eqs. (64, 65, and 66a–d) to proceed forward with the given step size, calculations can be made automatically. The step size can be varied along the solution path. Equations (65) yield the coefficients and Eq. (63) is the final solution.

One can see that, for a user, the information that is needed as inputs for a computer subroutine for the present method includes the right-hand side of Eq. (62), the initial conditions shown in Eqs. (67), and the step size.

For a second-order equation, corresponding to the jth subdivision,

$$y_j'' = f(y_j, y_j') \tag{69}$$

the solution is

$$y_j = A_j + B_j x + \tfrac{1}{2} a_j x^2, \; x_{j-1} \le x \le x_j \tag{70}$$

Equations (64, 67, and 68) are still valid, and Eqs. (65) and (66) are to be replaced, for the second-order system, by

$$A_j = A_{j-1} - \tfrac{1}{2} x_{j-1}^2 (a_{j-1} - a_j) \tag{71}$$

$$B_j = B_{j-1} + x_{j-1}(a_{j-1} - a_j) \tag{71a}$$

$$y_j|_{x=\bar{x}_j} = A_{j-1} - \tfrac{1}{2} x_{j-1}^2 (a_{j-1} - a_j) + [B_{j-1} + (a_{j-1} - a_j) x_{j-1}] \bar{x}_j + \tfrac{1}{2} a_j \bar{x}_j^2 \tag{72}$$

$$y_j'|_{x=\bar{x}_j} = B_{j-1} + x_{j-1}(a_{j-1} - a_j) + a_j \bar{x}_j \tag{72a}$$

Compared with the Runge-Kutta and some predictor-corrector methods, the derivations and equations in the present generalized method seem much simpler and direct. The second-order system has been programed as a subroutine and used on the first example. The result is shown in columns 6 and 7 of Table 1. It can be seen that for this particular problem, and using the routines available in the IBM subroutine package, the present generalized method compares favorably with the Runge-Kutta fourth-order method and Hamming's modified predictor-corrector method.

IV. Conclusions

Two versions of the present method have been described. Conceivably, the specialized approach presented in Sec. II produces a more accurate solution for the same step size, since its complementary solution is the exact solution of the linear part of the equation, and the remaining constant a_j is being used merely to represent the nonlinear part of the equation. In the generalized approach described in Sec. III, the constant a_j is extended to cover all the terms in the equation except the first term, which has the highest order. Consequently, the complementary solution can be standardized for general applications and the difference for different problems will be taken care of by the characteristic equation which is to be solved by numerical methods.

Based on the limited result from Table 1, one gets an impression that by comparison with the present method, the Runge-Kutta method and Hamming's modified predictor-corrector mehotd are more accurate when the step size is large, whereas the present method, including the generalized version, are superior in the ultimately achievable accuracy. If the impression is true, it would be an advantage in using the present method in computerized applications, because in machine computations, it is the ultimate achievable digital accuracy that limits the usefulness of a method, not the step size.

It might be of interest to mention that during the development of the method, several different ways were tried in the one-term solution to evaluate a_j in the characteristic equation, which is the equation that equates a_j to the nonlinear part of the differential equation of the jth interval. Of the alternatives examined, which included the obvious mean-error-squared-integration and another that satisfied the differential equation exactly at a terminal point instead of at the midpoint of the interval, the midpoint approach yields by far the most accurate solution. It prompts us to reflect that most numerical methods, and also the method of continuous analytical continuations, usually satisfy the differential equation at the terminal points but may not at points in between. It is possible that if the differential equation could be satisfied also at the midpoints of the intervals, the accuracy of some existing numerical methods might be improved significantly.

In conclusion, one may observe that the present method has the advantage of an approximate analytic method in having a continuous differentiable solution, yet it is a numerical method in the sense that numerical procedures are involved. The method seems to have the following merits: a) Its logic is direct, calculations are simple, and no high-order differentiations or finite-difference schemes are involved; b) the method is self-starting, step size can be varied and its accuracy is comparable to, or may be superior than that of other numerical methods, such as the Runge-Kutta and some predictor-corrector methods; c) certain error estimate is possible; d) programing of the generalized version is very simple and a subroutine for computer software is feasible.

A brief survey showed that the governing nonlinear differential equations associated with a number of problems that have been discussed in recent literature might be solved alternatively by the present method in a routine manner.[8–12]

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